

# A Song of Six Splatts

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‘The proteiform graph itself is a polyhedron of scripture.’

– James Joyce, ‘Finnegans Wake’

Many readers will no doubt have encountered Piet Hein’s famous ‘Soma Cube’, a puzzle consisting of the seven ‘irregular’ shapes which can be formed by combining up to four identical cubes, and from which a variety of structures, including a larger cube, can be made [1]. This puzzle relies on the intuitively obvious fact that identical cubes will tessellate to fill three-dimensional space. The cube is, however, by no means the only polyhedron with this property: [2] lists the other ‘symmetrical’ space-filling solids as the rhombic dodecahedron, the truncated octahedron and the tetrahedron with bevelled vertices, and this list is widely believed to be complete [3].

It is the truncated octahedron and puzzles derived from its space-filling property which we shall be considering in this article. For reasons too involved to go into here, we shall hereinafter term it the ‘splatt’. The splatt can be obtained from a regular octahedron by cutting off its vertices at the points of trisection of its edges. Thus it has eight hexagonal and six square faces, all with the same side length (see Figure 1). Around each hexagonal face, squares and hexagons alternate. Since each of its 24 vertices has the same appearance, with one square and two hexagonal faces meeting there, it is called, appropriately enough, an ‘Archimedean’ solid. The geometric properties of this shape are more fully discussed in [4], which also gives a net.

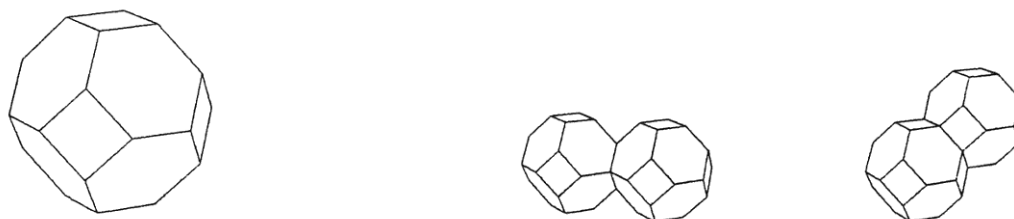


Figure 1. The truncated octahedron or ‘splatt’, and the two 2-splatts.

The splatt packs in a body-centred cubic lattice, which chemists will recognise as the crystal structure of caesium chloride. By considering the lattice as two interleaved cubic lattices, it may be seen that the volume of the splatt is exactly half that of the circumscribed cube whose faces include the square faces of the splatt. Each splatt in the packing has the same orientation, and so whenever adjacent splatts share a hexagonal face, there is a square face of one of them adjacent to each side of the common hexagon.

When  $n$  identical splatts are joined together by faces in such a way that they could form part of the space-filling packing, we call the resulting shape an  $n$ -splatt. Two  $n$ -splatts are to be considered equivalent if there is a rotation of 3-space which maps one onto the other. Trivially there is a unique 1-splatt; a little thought reveals that there are two distinct 2-splatts, one consisting of two splatts joined by square faces, the other of two splatts joined by hexagonal faces (see Figure 1). The essential uniqueness of the latter type of 2-splatt follows from the remark about hexagonal joins made above.

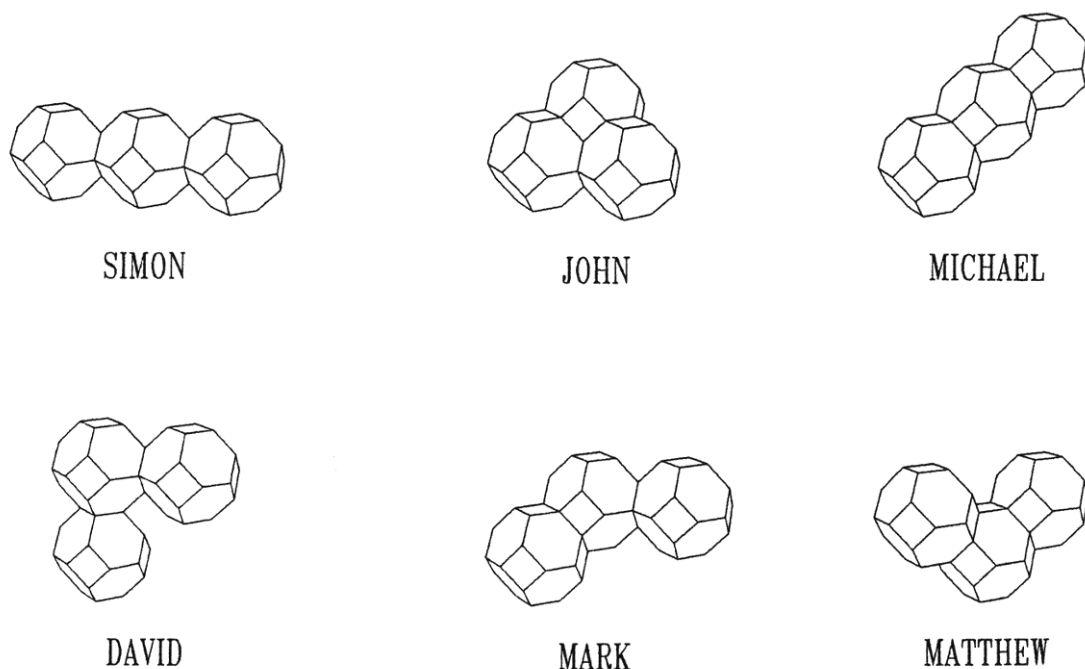


Figure 2. The six 3-splatts.

The reader may like to verify that adding another splatt can result in any of the six possibilities shown in Figure 2. This result contrasts with the corresponding result for combinations of three identical cubes, of which there are only two types. The first set of 3-splatts of which the authors are aware was constructed from apple pie cartons on Sunday 11th May 1986 at a meeting of the Puzzles and Games Ring. The 3-splatts were subsequently named in honour of the six people present on that occasion. When more than three splatts are joined, the phenomenon of chirality or handedness arises: there exist 4-splatts which cannot be rotated in 3-space to become their own mirror image. The authors believe there to be 44 4-splatts, 394 5-splatts and 4680 6-splatts, including both of each mirror-image pair, but do not yet know how many 7-splatts there are.

It is clear that if a puzzle akin to the Soma Cube were to be constructed from some set of  $n$ -splatts, the 4-, 5- or 6-splatts would yield an unwieldy number of pieces, whereas the 1- or 2-splatts would not sustain interest for long. The best compromise between triviality and overcomplexity is achieved by the set of 3-splatts; moreover they do not suffer from the disadvantage of having distinct mirror-image forms. It transpires that these afford the splattist ample opportunity to exercise his creative talents, for they give rise to a plethora of fascinating puzzles.

Readers are urged to construct their own sets of 3-splatts, for a well-made set will give hours of enjoyment. They may readily be made from cardboard: it is best to make the eighteen truncated octahedra individually and then to glue them together into the six pieces. A slightly more durable set may be fabricated from expanded polystyrene by cutting down the circumcubes with a hot wire. The authors have also tried using fibreglass with appropriate moulds.

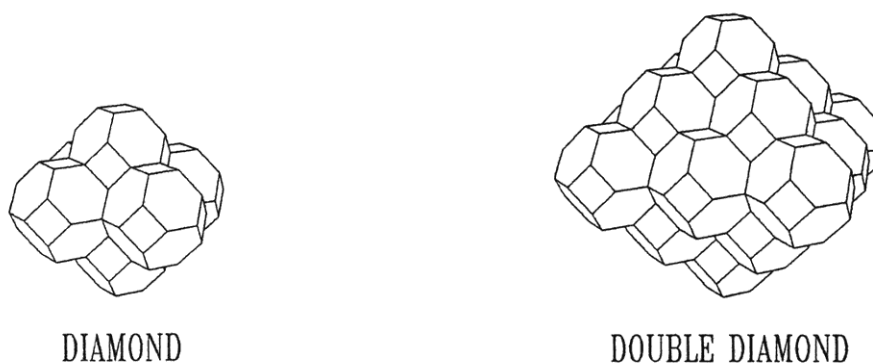


Figure 3. Two elementary 3-splatt puzzles.

An elementary puzzle for the dilettante splattist is to construct the diamond shape shown in Figure 3 from two of the 3-splatts. The resulting shape is an example of a 6-splatt with the interesting property that, if a vertex is marked for each constituent splatt, and a line is drawn between vertices corresponding to splatts which are joined at a face, a non-planar graph is formed. It is the only 6-splatt with this property, and no 5-splatts possess it. Once you have mastered that shape, try to use all the pieces to make the larger version pictured in Figure 3. Note that there is a central cavity in the shape of a 1-splatt. This is a relatively easy 3-splatt puzzle: it has 24 essentially different solutions.

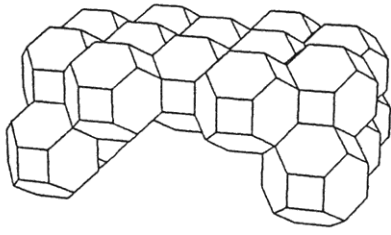
Further puzzles involving the complete set of 3-splatts are shown in Figures 4 and 5. All the shapes shown have symmetry, except ‘giraffe’, where the tail makes the figure asymmetric. There are no unexpected hidden cavities or projections, except in ‘ziggurat’ (for which we are grateful to Philip Belben), which has a single splatt missing from the middle of its base. Thanks are also due to Ian Stark for ‘drum’ and ‘triangles’, and to Simon James for ‘tortoise’. It is surprising fact that ‘bridge’ can actually be made to support itself in the middle!

To avoid hours of fruitless effort, we feel obliged to remark that ‘tower’ is impossible. We will give the proof here, as it is instructive and can be applied with success to other shapes which the reader may devise, but be unable to construct. Consider a colouring of the packing in two colours where splatts joined by squares are of the same colour, but those joined by hexagons are of opposite colours. The two colours can be identified with the two types of ion in the structure of caesium chloride mentioned above. With this colouring, the central vertical column of three splatts within the tower will be of one colour, and the other fifteen splatts will be of the other. So, if this structure were to be made from the six 3-splatts, at least three of them would need to be monochromatic, i.e. to have joins only along square faces. Of the six 3-splatts, however, only two, namely Simon and David, have this property. Thus the figure is impossible. Similarly any other structures with such a high ‘net charge’ will be impossible to construct.

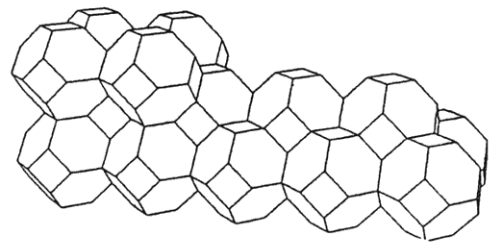
These are just a few examples of the large number of stunningly realistic shapes that can be formed with the set. Doubtless you will be able to find many more. If you come across any of especial note, we would be very interested to see them. We conclude with a problem: find the cuboid of least volume into which the set of 3-splatts may be packed.

## References

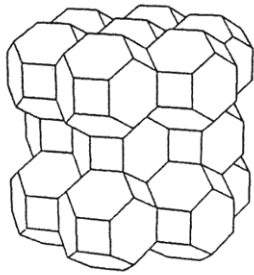
1. Martin Gardner, ‘More Mathematical Puzzles and Diversions’, Penguin (1963) pp 50–9
2. J. E. Drummond, ‘Space filling with identical symmetrical solids’ *Math. Gaz.* 68 (1984) pp 104–6
3. Branko Gruenbaum and G. C. Shephard, ‘Space filling with identical symmetrical solids’, *Math. Gaz.* 69 (1985) pp 117–20
4. H. M. Cundy and A. P. Rollett, ‘Mathematical models’, Oxford (1974) p 104



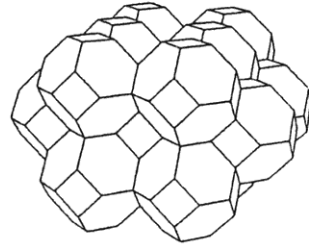
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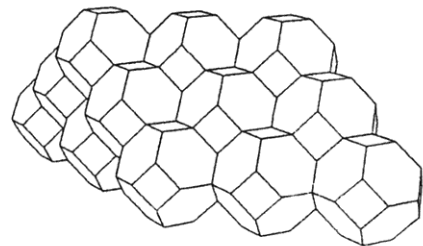
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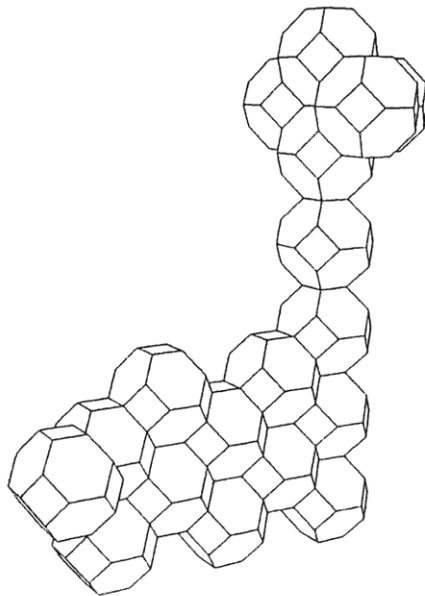
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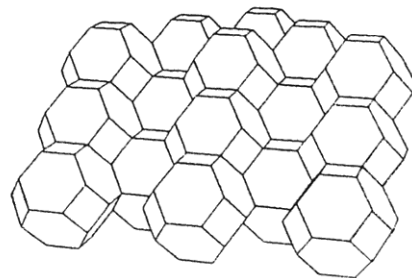
EGGBOXES



LOZENGE

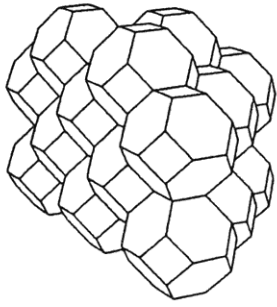


GIRAFFE

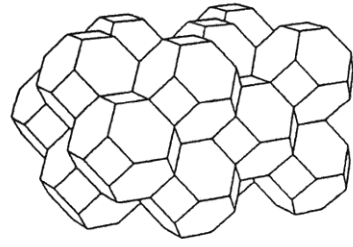


NEW COURT CHRIST'S

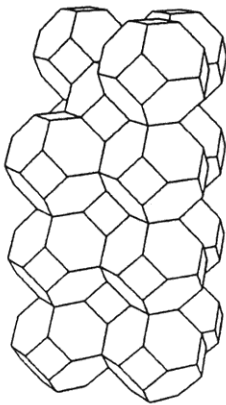
Figure 4. A collection of 3-splatt puzzles.



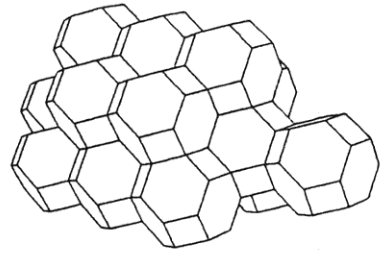
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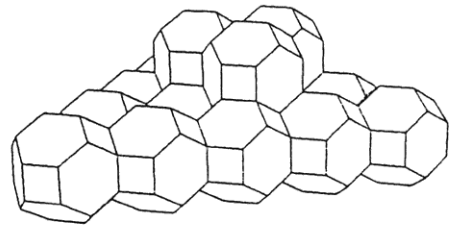
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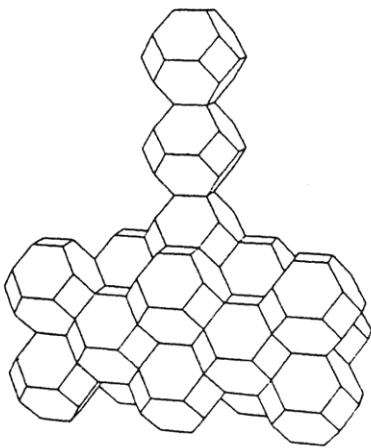
TOWER



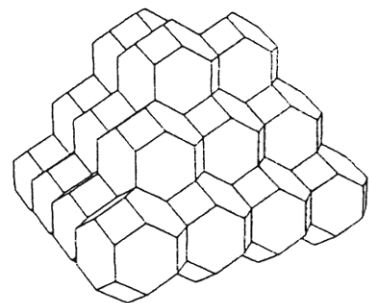
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ZIGGURAT

Figure 5. Further 3-splatt puzzles.